SIAM J. MATH. ANAL.
Vol. 14, No. 2, March 1983

0036-1410/83/1402-0013 $01.25/0

TOTAL POSITIVITY OF MEAN VALUES AND HYPERGEOMETRIC FUNCTIONS*

B. C. CARLSON† AND JOHN L. GUSTAFSON†

Abstract. The weighted power mean of two positive variables is strictly totally positive (STP) if its order t satisfies $-\infty < t < 0$ and its reciprocal is STP if $0 < t < \infty$. The reciprocals of the logarithmic mean, Gauss's arithmetic-geometric mean, and the Schwab-Borchardt mean are STP. The hypergeometric R-function $R_{-\alpha}(\beta, \beta'; x, y)$, $x, y > 0$, which is equivalent to ${}_2F_1$ with argument $1 - x/y$, is STP if α, β, β', and $\beta + \beta' - \alpha$ are positive. With weaker restrictions this function is represented in a new way as a convolution. Higher order positivity is discussed for some other hypergeometric functions, including incomplete elliptic integrals.

1. Introduction. A real-valued function $f(x, y)$ of two real variables is said to be strictly totally positive (STP) on its domain of definition if every $n \times n$ determinant with elements $f(x_i, y_j)$, where $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$, is strictly positive for every $n = 1, 2, \cdots$. If the determinants are strictly positive for $n = 1, 2, \cdots, r$, then f is said to be strictly positive of order r (SP_r). The principal reference for the subject is Karlin [6], who writes STP_r in place of SP_r and sometimes STP_∞ for STP. Many applications to statistics, mechanics, and differential equations arise from the circumstance that a totally positive function is the kernel of a variation-diminishing transform.

We refer to [6] or [7, Chap. 18] for more precise statements and proofs of several basic facts:

(1.1) e^{xy} is STP for x, y real [6, pp. 15–16].

(1.2) If both g and h are strictly increasing functions, or if both are strictly decreasing, and if $F(x, y) = f(g(x), h(y))$, then F is SP, if f is SP_r [6, p. 18].

(1.3) If g and h are strictly positive functions, and if $F(x, y) = g(x) f(x, y) h(y)$, then F *is* SP_r [6, p. 18].

(1.4) If $f(x, y) = \int_Z g(x, z)h(z, y))\, d\sigma(z)$, where σ is a positive σ-finite measure on Z and the integral converges absolutely, then f is SP_r on $X \times Y$ if g is SP_r on $X \times Z$ and h is SP_r on $Z \times Y$ [6, pp. 16–17].

To these four rules we add two more:

(1.5) If (1.4) is modified so that either

$$\frac{1}{f(x,y)} = \int_Z \frac{h(z,y)}{g(x,y)}\, d\sigma(z) \text{ or } f(x,y) = \int_Z \frac{d\sigma(z)}{g(x,z)h(z,y)},$$

then f is SP_2 if g and h are SP_2. This follows from [6, Eq. (2.5)] and the observation that $a_{11}, a_{12}, a_{21}, a_{22} > 0$ implies that the 2×2 determinant with elements a_{ij} is strictly positive if and only if the 2×2 determinant with elements $1/a_{ij}$ is strictly negative.

(1.6) If $a > 0$ then $(x + y)^{-a}$ is STP for $x, y > 0$.

Apparently (1.6) is new except for the case $a = 1$ [6, pp. 149–150], which dates back to Cauchy and demonstrates that all minors of the Hilbert matrix are positive. The

*Received by the editors October 19, 1981.

†Ames Laboratory and Department of Mathematics, Iowa State University, Ames, Iowa 50011. Ames Laboratory is operated for the U. S. Department of Energy by Iowa State University under contract W-7405-Eng-82. This research was supported by the Director of Energy Research, Office of Basic Energy Sciences.

proof of the general case follows from the integral representation of the gamma function [2, Ex. 3.2-3],

$$(x+y)^{-a}\Gamma(a) = \int_0^\infty t^{a-1} e^{-(x+y)t}\,dt,$$
$$(x+y)^{-a} = \int_{-\infty}^0 e^{xz} e^{zy}\,d\sigma(z),$$

where $d\sigma = (-z)^{a-1} dz / \Gamma(a)$. The proof is completed by using (1.1) and (1.4).

2. Power means. The weighted power mean [4, p. 13] of order t is defined by

(2.1) $$M_t(x,y) = \left[wx^t + (1-w)y^t\right]^{1/t}, \qquad t \neq 0$$

where $x, y > 0$ and $0 < w < 1$.

THEOREM 2.1. *If* $0 < t < \infty$, *then* $1/M_t(x, y)$ *is STP for* $x, y > 0$. *If* $-\infty < t < 0$ *then* $M_t(x, y)$ *is STP for* $x, y > 0$.

Proof. It follows from (1.6) and (1.2) that $[wx^t + (1-w)\,y^t]^{-a}$ is STP if $a > 0$ and $t \neq 0$. Assuming $0 < t < \infty$ and putting $a = 1/t$, we conclude that $1/M_t(x, y)$ is STP. If $-\infty < t < 0$ we put $a = -1/t$.

Note that the geometric mean, $M_0(x, y) = x^w y^{1-w}$, is not STP because the rows of the relevant determinants are proportional. The possibility of proportional rows likewise keeps M_∞ and $M_{-\infty}$ [4, p. 15] from being STP, although the determinants are nonnegative.

If $a > 0$ and $c \geq 0$, $(x + y + c)^{-a}$ is STP for $x, y > 0$ by (1.6) and (1.2). Hence the weighted power mean of several variables, $[\Sigma w_i x_i^t]^{1/t}$, has the positivity properties of Theorem 2.1 in any two of the variables if the others are held fixed.

3. Iterative means. If $x, y > 0$ let $x_0 = x$ and $y_0 = y$ and consider three separate iterative processes in which x_n and y_n approach a common limit as $n \to \infty$:

(3.1) $$x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}(x_n y_n)^{1/2}, \qquad y_{n+1} = \frac{1}{2}y_n + \frac{1}{2}(x_n y_n)^{1/2}, \qquad x_n, y_n \to L(x,y),$$

(3.2) $$x_{n+1} = (x_n + y_n), \qquad y_{n+1} = (x_n y_n)^{1/2}, \qquad x_n, y_n \to M(x,y),$$

(3.3) $$x_{n+1} = (x_n + y_n), \qquad y_{n+1} = (x_{n+1} y_n)^{1/2}, \qquad x_n, y_n \to S(x,y).$$

Here L is the logarithmic mean, M is Gauss's arithmetic-geometric mean, and S is the Schwab-Borchardt mean[1]. The reciprocal of each has an integral representation [1]:

(3.4) $$\frac{1}{L(x,y)} = R_{-1}(1,1;x,y) = \frac{\ln x - \ln y}{x - y},$$

(3.5) $$\frac{1}{M(x,y)} = R_{-1/2}\left(\frac{1}{2},\frac{1}{2};x^2,y^2\right),$$

(3.6) $$\frac{1}{S(x,y)} = R_{-1/2}\left(\frac{1}{2},1;x^2,y^2\right) = \begin{cases} (y^2 - x^2)^{-1/2}\arccos(x/y), & x < y, \\ (x^2 - y^2)^{-1/2}\operatorname{arccosh}(x/y), & x > y, \end{cases}$$

[1]The iterative process converging to S was proposed but not published by Gauss in 1800 (for more details see [1]). Schwab [9, pp. 103–107] published it in 1813 and Borchardt in 1880. We thank Professor I. J. Schoenberg for reference [9].

where

$$R_{-\alpha}\left(\beta,\beta';x,y\right)=\int_0^\infty (x+z)^{-\beta}\left(z+y\right)^{-\beta'}d\sigma\left(z\right),$$

(3.7)

$$d\sigma\left(z\right)=\frac{\Gamma\left(\beta+\beta'\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta+\beta'-\alpha\right)}z^{\beta+\beta'-\alpha-1}dz,\qquad 0<\alpha<\beta+\beta'.$$

It follows from (1.4) and (1.6) that $R_{-\alpha}(\beta, \beta'; x, y)$ is STP for $x, y > 0$ provided $\beta, \beta' > 0$ and $0 < \alpha < \beta + \beta'$. Use of (1.2) completes the proof of the following theorem:

THEOREM 3.1. *The reciprocal means* $1/L(x, y)$, $1/M(x, y)$, *and* $1/S(x, y)$ *are STP for* $x, y > 0$.

The means M and S are the best-known members of a family of twelve iterative means $L_{ij}(x, y)$ constructed by letting

(3.8) $$x_{n+1}=f_i(x_n, y_n),\qquad y_{n+1}=f_j(x_n, y_n),\qquad i\neq j,$$

where

$$f_1\left(x,y\right)=\frac{1}{2}\left(x+y\right),\qquad f_2\left(x,y\right)=\left(xy\right)^{1/2},$$

(3.9)

$$f_3\left(x,y\right)=\left(x\frac{x+y}{2}\right)^{1/2},\qquad f_4\left(x,y\right)=\left(y\frac{x+y}{2}\right)^{1/2}.$$

For each of the twelve choices of i and j, $i\neq j$, the common limit of x_n and y_n as $n\to\infty$ is $L_{ij}(x, y)$. For example, the Schwab-Borchardt mean S is L_{14}. In each case a suitable negative power ($-1/2$ or -1 or -2) of L_{ij} (see [1]) is an R-function (3.7) with α, β, β' such that it is STP. The mean L also is essentially a member of this family, as one sees by replacing each variable in (3.1) by its square.

4. Hypergeometric functions. The R-function (3.7) is a homogeneous variant of Gauss's hypergeometric function [2, §5.9]:

(4.1) $$R_{-\alpha}\left(\beta,\beta';x,y\right)=y^{-\alpha}\,{}_2F_1\left(\alpha,\beta,\beta+\beta';1-\frac{x}{y}\right)$$

If b is a k-tuple of real numbers and x is a k-tuple of positive numbers, an extension of (3.7) to several variables is [2, (6.8-6)]

$$R_{-a}\left(b,x\right)=\int_0^\infty\prod_{i=1}^{k}\left(x_i+z\right)^{-b_i}d\sigma\left(z\right),$$

(4.2)

$$d\sigma\left(z\right)=\frac{\Gamma\left(a+a'\right)}{\Gamma\left(a\right)\Gamma\left(a'\right)}z^{a'-1}dz,\qquad a'=\sum_{i=1}^{k}b_i-a_i,\qquad a>0,\quad a'=0$$

The R-function has other representations that define it when a and a' are not positive.

THEOREM 4.1. *Let* $a, a', b_1, \cdots, b_k$ *be real numbers and assume* $a+a'=\Sigma_{i=1}^{k}b_i$ *and* $aa'b_1\cdots b_k\neq 0$. *Let* $x_i>0$, $i=1,\cdots,k$. *For some* i *and* j *consider* $R_{-a}(b, x)$ *as a function of* x *and* x_j, *all other components of* x *being fixed; i.e., define* $f(x_i, x_j)=[(x_i, x_j)\mapsto R_{-a}(b, x)]$. *If* $k\geq 2$ *and* $a, a', b_i, b_j>0$, *then* f *is STP. If* $k=2$ *and exactly one of* a, a', b_1, b_2 *is negative, then* $1/f$ *is* SP_2. *If* $k>2$ *and* $a, a'>0$, *then* $1/f$ *is* SP_2 *if* $b_ib_j<0$ *while* f *is* SP_2 *if* $b_i<0$ *and* $b_j<0$.

Proof. In those parts of the theorem which assume $a, a' > 0$, we may use (4.2) and define a sigma-finite measure

$$d\sigma_1(z) = \prod_{m \neq i,j} \left(x_m + z\right)^{-b_m} d\sigma(z).$$

If $b_i, b_j > 0$ then (1.6) and (1.4) imply that f is STP. If $b_i < 0$ then $\left(x_i + z\right)^{-b_i}$ is the reciprocal of a function that is STP and therefore SP_2. Hence the last sentence of the theorem follows from (1.5), as does the next to last sentence in case exactly one of b_1 and b_2 are positive, follows from [2, (5.9-20)] and (1.3).

Theorem 4.1 has interesting applications to elliptic integrals. For example, the perimeter of an ellipse [2, (9.4-5)] with semiaxes α and β is $P(\alpha, \beta) = 2\pi R_{1/2}(½, ½, \alpha^2, \beta^2)$, and hence $1/P(\alpha, \beta)$ is SP_2 for $\alpha, \beta > 0$. The symmetric incomplete integrals of the first and third kinds [3],

$$R_F(x,y,z) = R_{-1/2}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};x,y,z\right), \qquad R_J(x,y,z,p) = R_{-1/2}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},1;x,y,z,p\right),$$

where $x, y, z, p > 0$, are STP in any two variables when the others are fixed. We may choose $z = 1$ by homogeneity and tabulate $R_F(x, y, 1)$ with rows and columns of the table labeled by increasing values of x and y, respectively. If the table is regarded as a matrix, all its minors are strictly positive. Similar remarks apply to the integral of the second kind, $R_D(x, y, z) = R_J(x, y, z, z) = R_{-3/2}(½, ½, ³⁄₂, x, y, z)$

Theorem 4.1 implies that $1/R_t(\beta, \beta'; x, y)$ is SP_2 for $x, y > 0$ provided $\beta, \beta' > 0$ and either $t > 0$ or $t < -\beta - \beta'$. We ask now whether the SP_2 property can be strengthened to STP or at least SP_r for some $r > 2$. Because of [2, (5.9-21)] and (1.3), $1/R_t(\beta, \beta'; x, y)$ is SP_r if and only if $1/R_{-\beta-\beta'-t}(\beta, \beta'; x, y)$ is SP_r. Hence it suffices to consider the case $t > 0$.

If $\beta, \beta' > 0$, it is not hard to show that $1/R_t(\beta, \beta'; x, y)$ is STP for $x, y > 0$ in certain special and limiting cases. If $t = 1$ we use [2, (6.2-2)]. For any $t > 0$, as $\beta + \beta'$ tends to 0 or ∞ with β/β' fixed, we use [2, (6.2-17), (6.2-18)]. (The cited equations are valid also for nonintegral n.) Some additional special cases in which $1/R_t$ is STP if $t > 0$ will be exhibited in §5.

Nevertheless, a numerical example shows that $1/R_2(½, ½; x, y)$ is not SP_3. If $(x_1, x_2, x_3) = (1, 2, 3)$ and $(y_1, y_2, y_3) = (100, 200, 300)$, the 3×3 determinant with elements $1/R_2(½, ½; x_i, y_j)$ has the value -1.7×10^{-20}. More generally a complicated algebraic expression for the 3×3 determinant with elements $1/R_2(\beta, \beta'; x_i, y_j)$ shows that the determinant will be negative for fixed positive $\beta < 1$ if x_3/y_1 (or y_3/x_1) is sufficiently small.

We conclude that if $t > 0$ or $t < -\beta - \beta'$, then $1/R_t$ is sometimes STP and sometimes not even SP_3 but always SP_2 if $\beta, \beta' > 0$. Some further examples in which it is or is not STP will be discussed in the next section by using the properties of Pólya frequency functions.

Since the weighted power mean (2.1) of order t is the limit as $c \to 0+$ of the hypergeometric mean $[R_t(cw, c - cw; x, y)]^{1/t}$, it is natural to ask whether the reciprocal of the latter is STP if $c > 0$ and $t > -c$. In general it is not. For instance, if $(x_1, x_2, x_3) = (1, 2, 3)$ and $(y_1, y_2, y_3) = (100, 200, 300)$, the 3×3 determinant with elements $1/R_2(½, ½; x_i, y_j)^{1/2}$ has the value -8.1×10^{-15}.

5. Pólya frequency functions. A measurable real-valued function f defined on the real line is called a strict Pólya frequency function (SPF) if $f(x - y)$ is STP. (Some authors require f to be integrable, but if f is SPF then $e^{cx} f(x)$ is integrable for suitable real c [8, p. 341].) If $f(x-y)$ is SP_r then f is called SPF_r. A function is SPF_2 if and only if it is strictly log-concave on the real line [8, p. 337].

For example, if $\beta, \beta' > 0$ and $0 < \alpha < \beta + \beta'$, then $R_{-\alpha}(\beta, \beta'; e^{2x}, e^{2y})$ is STP for real x and y by Theorem 4.1 and (1.2). Since $R_{-\alpha}$ is homogeneous of degree $-\alpha$, we have

$$R_{-\alpha}(\beta, \beta'; e^{2x}, e^{2y}) = e^{-\alpha x} e^{-\alpha y} R_{-\alpha}(\beta, \beta'; e^{x-y}, e^{y-x}).$$

It follows by (1.3) that $R_{-\alpha}(\beta, \beta'; e^{x}, e^{-x})$ is SPF.

For another example, the Gegenbauer polynomial [2, (6.7-21)] of degree n is

$$C_n^{\nu}(\cosh x) = \frac{\Gamma(2\nu+n)}{\Gamma(2\nu)\Gamma(n+1)} R_n(\nu, \nu; e^x, e^{-x}) \tag{5.1}$$

If $\nu > 0$ and $n = 1, 2, 3, \cdots$, it follows from Theorem 4.1 that $1/C_n^{\nu}(\cosh x)$ is SPF_2 and $C_n^{\nu}(\cosh x)$ is strictly log-convex. The same is true for the Gegenbauer function defined by (5.1) with any real $n > 0$ and $\nu > 0$.

To see whether $1/C_n^{\nu}$ is SPF, we shall use a theorem of Schoenberg [8, p.349] with strictness conditions added by Karlin [6, p. 357]. Only an abridged version of the theorem will be needed. A measurable real-valued function f defined on the real line is SPF if its bilateral Laplace transform exists in an open strip containing the imaginary axis and has the form

$$\int_{-\infty}^{\infty} e^{-sx} f(x)dx = \frac{1}{\varphi(s)}, \qquad \varphi(s) = Ce^{\delta s}\prod_{i=1}^{\infty}(1+a_i s)e^{-a_i s}, \tag{5.2}$$

where $C > 0$, the a_i and δ are real, Σa_i^2 converges, and $\Sigma|a_i|$ diverges. Conversely, f is not SPF unless the reciprocal of its bilateral Laplace transform is entire.

For example, if $\beta, \beta' > 0$, $-\alpha < \operatorname{Re} s < \alpha$, and $\alpha - 2\beta < \operatorname{Re} s < 2\beta - \alpha$, then

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{-sx} R_{-a}(\beta, \beta'; e^x, e^{-x})dx \\ &= \frac{\Gamma(\beta+\beta')\Gamma\left(\frac{\alpha+s}{2}\right)\Gamma\left(\frac{\alpha-s}{2}\right)\Gamma\left(\frac{2\beta-\alpha+s}{2}\right)\Gamma\left(\frac{2\beta'-\alpha-s}{2}\right)}{2\Gamma(\beta)\Gamma(\beta')\Gamma(\alpha)\Gamma(\beta+\beta'-\alpha)},\end{aligned} \tag{5.3}$$

as one finds by taking e^{-x} as a new integration variable to obtain a Mellin transform, substituting (3.7), and changing the order of integration. The representation of Γ by an infinite product shows that (5.3) has the form (5.2). This was expected, since the conditions of validity imply $0 < \alpha < \beta + \beta'$.

Since the product of the Laplace transforms of two functions is the transform of their convolution, (5.3) suggests a new way of writing the hypergeometric function (4.1) as a convolution:

$$R_{-a}(\beta, \beta'; e^x, e^{-x}) = \frac{2^{1-\beta-\beta'}}{B(\beta, \beta')}\int_{-\infty}^{\infty} \operatorname{sech}^{\alpha}(x-t) e^{(\beta+\beta'-\alpha)} dt, \tag{5.4}$$

where $|\operatorname{Im} x| < \pi/2$, $\operatorname{Re}\beta > 0$, and $\operatorname{Re}\beta' > 0$. These conditions of validity can be verified by putting $e^{2t} = (1-u)/u$ to obtain Euler's representation. Equation (5.4) is particularly attractive if β and β' are equal, as they are for Legendre and Gegenbauer functions [2, §6.8].

We can now investigate further the higher order positivity of $1/R_t$, $t>0$. For example,

$$\int_{-\infty}^{\infty}\frac{e^{-sx}\,dx}{R_t\left(1,1;e^x,e^{-x}\right)}=\frac{\pi\sin\left(\dfrac{\pi}{t+1}\right)}{2\sin\left(\dfrac{\pi}{2}\dfrac{t+s}{t+1}\right)\sin\left(\dfrac{\pi}{2}\dfrac{t-s}{t+1}\right)},\qquad -t<\operatorname{Re}s<t. \tag{5.5}$$

This result follows from (5.3): observe that [2, Ex. 5.9-13]

$$\frac{1}{R_t\left(1,1;e^x,e^{-x}\right)}=\frac{(t+1)\sinh x}{\sinh\left[(t+1)x\right]}=R_{-t/(t+1)}\left(1,1;e^y,e^{-y}\right),\qquad y=(t+1)x.$$

The representation of the sine function by an infinite product shows that (5.5) has the form (5.2). Hence $1/R_t(1,1;e^x,e^{-x})$, $t>0$, is SPF and $1/R_t(1,1;x,y)$, $t>0$ is STP for $x,y>0$.

Another example, in which the Laplace transform can be evaluated by using [2, Ex. 6.10-12, (4.2-4)] after taking e^{-x} as a new variable of integration, is

$$\int_{-\infty}^{\infty}\frac{e^{-sx}\,dx}{R_t\left(\frac{1}{2}-t,\frac{1}{2}-t;e^x,e^{-x}\right)}=\frac{2^{2t}\,\Gamma(t+s)\Gamma(t-s)}{\Gamma(2t)}, \tag{5.6}$$

where $-t<\operatorname{Re}s<t$ and $t\neq 1,2,3,\cdots$. Since this has the form (5.2), the condition $0<t\neq 1,2,3,\cdots$ ensures that $1/R_t(\frac12-t,\frac12-t,e^x,e^{-x})$ is SPF and $1/R_t(\frac12-t,\frac12-t,x,y)$ is STP for $x,y>0$. The same is true of $1/R_{t-1}(\frac12-t,\frac12-t,x,y)$ with the same conditions on t,x,y (see [2, Ex. 6.10-12]).

Despite the preceding special cases (as well as the cases mentioned near the end of §4) in which $1/R_t$, $t>0$, is STP, a final example suggests that this state of affairs may be the exception rather than the rule. If β,β', are real and $\beta\beta'(\beta+\beta'+1)>0$, then [2, (6.2-4)] yields

$$\int_{-\infty}^{\infty}\frac{e^{-sx}\,dx}{R_2\left(\beta,\beta';e^x,e^{-x}\right)}=\frac{\pi}{2}(\beta+\beta')(\tan\theta)\left[\frac{\beta(\beta+1)}{\beta'(\beta'+1)}\right]^{s/4}\frac{\sin(s\theta/2)}{\sin(s\pi/2)}, \tag{5.7}$$

where $-2<\operatorname{Re}s<2$, $0<\theta<\pi/2$, and $\tan\theta=[(\beta+\beta'+1)/\beta\beta']^{1/2}$. If $\pi/\theta=3,4,5,\cdots$, all zeros of $\sin(s\theta/2)$ are cancelled by zeros of $\sin(s\pi/2)$. Then (5.7) has the form (5.2) and $1/R_2(\beta,\beta';e^x,e^{-x})$ is SPF. (The case $\beta=\beta'=1$ coincides with the case $t=2$ of (5.5).) In particular, by (5.1), $1/C_2^{\nu}(\cosh x)$ is SPF if $\nu/(\nu+1)=\cos(\pi/m)$, $m=3,4,5,\cdots$. On the other hand, if $0<\theta<\pi/2$ but θ does not have one of the listed values, the reciprocal of the Laplace transform is not entire and $1/R_2(\beta,\beta';x,y)$ is not STP. The numerical example in §4 shows that it is not always in SP_3.

Other interesting examples of sign regularity properties of hypergeometric functions are contained in [10].

REFERENCES

[1] B. C. CARLSON, *Algorithms involving arithmetic and geometric means*, Amer. Math. Monthly, 78 (1971), pp. 496–505.
[2] _______, *Special Functions of Applied Mathematics*, Academic Press, New York, 1977.
[3] _______, *Computing elliptic integrals by duplication,* Numer. Math. 33 (1979), pp. 1–16.
[4] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, 2nd ed., Cambridge Univ. Press, Cambridge, 1952

[5] I.I. HIRSCHMAN AND D. V. WIDDER, *The Convolution Transform*, Princeton Univ. Press, Princeton NJ, 1955.

[6] S. KARLINE, *Total Positivity*, Stanford Univ. Press, Stanford, CA, 1968.

[7] A. W. MARSHALL AND L. OLKIN, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.

[8] I. J. SCHOENBERG. *On Pólya frequency functions*. I. *The totally positive functions an their Laplace transforms*, J. d'Anal. Math, 1 (1951), pp. 331–374.

[9] J. SCHWAB, *Élémens de géométrie*, vol. 1, C.-J. Hissette, Nancy, 1813.

[10] S. KARLIN, *Sign regularity properties of classical orthogonal polynomials*, in Orthogonal Polynomials and Their Continuous Analogues, D. T. Haimo, ed., Southern Illinois Univ. Press, Carbondale, 1968, pp. 55–74.